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CONVEXITY OF ELLIPTICALLY CONTOURED DISTRIBUTIONS  
WITH APPLICATIONS

BY

S. IYENGAR and Y. L. TONG

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CONVEXITY OF ELLIPTICALLY CONTOURED  
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1. Introduction.

The problem of evaluating multivariate probabilities arises in many areas of statistics, such as in the construction of confidence regions for the mean vector or regression parameters in a general linear model, the determination of critical regions for certain tests, multiple comparisons, reliability. etc. Because these probabilities are hard to evaluate numerically, a thorough study of the nature of, say, the distribution function is needed to suggest good approximations and inequalities. For example, tables for selected values of parameters are available, and one usually interpolates linearly to get an approximation; if we know that the tabulated function is convex or concave, we also get a bound for the quantity of interest.

It is well known that the cumulative and rectangular probabilities for elliptically contoured distributions are increasing functions of the correlations (see Tong [8] for more precise statements), but the rate of increase has not yet been studied. In this paper, we prove some results concerning the convexity in correlations of the distribution functions of random vectors with elliptically contoured distributions, and their absolute values. Sections 2 and 3 concentrate on the bivariate case, and Sections 4 and 5 provide some applications. The last section contains extensions to the multivariate case; the results here are not as strong as for the bivariate case, as the computations are much harder.

## 2. The Bivariate Normal.

The following identity due to Plackett [6] facilitates the computations that follow.

Lemma 1. Let  $\phi_n(x; \mathbb{A})$  be the multivariate normal density with mean zero and variances 1. If  $\mathbb{A} = (\rho_{ij})$ , then

$$(1) \quad \frac{\partial \phi}{\partial \rho_{ij}} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}.$$

When  $n=2$ , we write  $\phi(x; \rho)$  instead of  $\phi_2(x; \mathbb{A})$ . Identity (1) gives rise to natural conditions on partial correlations for certain results to be true. Now let

$$(2) \quad F_2(x; \rho) = P(X_1 \leq x_1, X_2 \leq x_2)$$

and

$$G_2(x; \rho) = P(|X_1| \leq x_1, |X_2| \leq x_2)$$

where  $X = (X_1, X_2)'$  has density  $\phi(x; \rho)$ . Then we have

Theorem 1.  $F_2$  is a convex (concave) function of  $\rho$  in an interval  $I$  if and only if  $h(x; \rho) = (\rho x_1 - x_2)(\rho x_2 - x_1) + \rho(1 - \rho^2) \geq (<) 0$  holds for  $\rho \in I$ .  $G_2$  is a convex (concave) function of  $\rho \in I$  if and only if

$$k(x; \rho) = (\rho x_1 + x_2)(\rho x_2 - x_1) - \rho(1 - \rho^2) + e^{\frac{2\rho x_1 x_2}{1 - \rho^2}} (\rho(1 - \rho^2) + (\rho x_1 - x_2)(\rho x_2 - x_1))$$

$$\geq (<) 0,$$

for  $\rho \in I$ .

Proof. Using (1) and doing the following interchange of differentiation and integration, we get

$$\frac{\partial}{\partial \rho} F_2 = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \frac{\partial^2}{\partial t_1 \partial t_2} \phi(t; \rho) dt = \phi(x; \rho) ,$$

So that

$$\frac{\partial^2}{\partial \rho^2} F_2 = \frac{1}{(1-\rho^2)^2} \phi(x; \rho) h(x; \rho) .$$

A similar computation yields

$$\frac{\partial^2}{\partial \rho^2} G_2 = \frac{2}{(1-\rho^2)^2} \phi(x_1, -x_2; \rho) k(x; \rho) .$$

Now the theorem is immediate.

More helpful, perhaps, is the following:

Corollary 1. Let

$$\frac{x_1}{x_2} \wedge \frac{x_2}{x_1} = m \quad \text{and} \quad \frac{x_1}{x_2} \vee \frac{x_2}{x_1} = M .$$

If  $x_1 x_2 > 0$ ,  $F_2$  is convex in at least  $(0, m)$ ; if  $x_1 x_2 < 0$ ,  $F_2$  is concave in at least  $(M, 0)$ .  $G_2$  is convex in at least  $(-m, m)$ .

In particular, if  $c = x_1 = x_2$  then  $F_2$  is convex in  $(0, 1)$  and when  $c > 0$ ,  $G_2$  is convex in  $(-1, 1)$ . For  $F_2$  we can get even more detailed information: if  $|c| \geq \sqrt{2} - 1$ ,  $F_2$  is convex in  $(-1, 1)$  since in that case  $h$  is positive for all  $\rho$ . In the next section, we will generalize the above



theorem and corollary to arbitrary elliptically contoured distributions, but the detailed information in this paragraph is much harder to get.

Corollary 1 shows that the interval of convexity depends on how spread out  $x_1$  and  $x_2$  are. A more precise statement of this uses the concept of majorization; for a full account, see Marshall and Olkin [5].

Theorem 2. Let  $x, y \in \mathbb{R}^2$ . If  $F_2(x; \rho)$  is a convex (concave) function of  $\rho \in I$ , and if  $x \succ y$  ( $y \succ x$ ), then  $F_2(y; \rho)$  is also convex (concave) in  $I$ .

Proof. Suppose that  $F_2(x; \rho) \geq 0$  and  $x \succ y$ . Note that  $h(x; \rho)$  in Theorem 1 is a Schur concave function of  $x$ . Thus  $h(y; \rho) \geq h(x; \rho) \geq 0$ , and  $\frac{\partial^2}{\partial \rho^2} F_2(y; \rho) = \frac{1}{(1-\rho^2)^2} \phi(y; \rho) h(y; \rho) \geq 0$ . The other case is similar.

Corollary 1 also shows that the convexity of  $G_2$  depends upon the spread of  $x_1$  and  $x_2$ . Indeed, we have ample numerical evidence to show that Theorem 2 also applies to  $G_2$ ; however, we do not yet have an analytical proof of it, as the algebra is quite cumbersome.

### 3. Bivariate Elliptically Contoured Distributions.

Some of the above results can be extended to all elliptically contoured distributions. Recall that a random vector  $X \in \mathbb{R}^n$  is elliptically contoured if there is a  $\mu$  for which  $(X - \mu)$  has characteristic function  $\hat{f}(t) = \phi(t' \Sigma^{-1} t)$  where  $\phi: [0, \infty) \rightarrow \mathbb{R}$ ; when  $X$  has a density, it is of the form  $|\Sigma|^{-1/2} g(x' \Sigma^{-1} x)$ , where  $\int_0^\infty x^{n-1} g(x^2) dx < \infty$ . A representation theorem of Cambanis, et al. [1] says that this is equivalent to saying that

$$(3) \quad X \stackrel{d}{=} \mu + R \mathbb{I}^{1/2} U^{(n)}$$

where  $R \geq 0$  is independent of  $U^{(n)}$ ,  $U^{(n)}$  is uniformly distributed on the unit sphere, and  $\mathbb{I}^{1/2}$  is the symmetric square root of  $\mathbb{I}$ .

To start our investigation, we need a generalization of Plackett's identity. Joag-dev, et al. [4] proposed the following: if  $g$  is a  $C^1$  function, then

$$(4) \quad \frac{\partial f}{\partial \rho_{ij}} = - \frac{\partial}{\partial x_j} \left( \sum_{k=1}^n \rho^{ik} x_k \right) f$$

where  $\mathbb{I}^{-1} = (\rho^{ij})$  and  $f(x; \mathbb{I}) = |\mathbb{I}|^{-1/2} g(x' \mathbb{I}^{-1} x)$ . We propose another generalization: suppose  $X$  has density

$$(5) \quad f_{n,p}(x; \mathbb{I}, \beta) = \frac{2 |\mathbb{I}|^{-1/2} \beta^{p + \frac{n}{2}}}{s_n \Gamma(\frac{n}{2} + p)} (x' \mathbb{I}^{-1} x) \exp(-\beta x' \mathbb{I}^{-1} x),$$

where  $s_n$  is the surface area of the unit ball in  $\mathbb{R}^n$ . Then, we have the following lemma:

Lemma 2. If  $X$  has density  $f_{n,p}$ , then its characteristic function is  $\phi_{n,p}(t' \mathbb{I} t; \beta)$  where

$$\phi_{n,p}(u; \beta) = e^{-u/4\beta} \sum_{k=0}^p \binom{p}{k} \prod_{i=0}^{k-1} \left(\frac{n}{2} + i\right)^{-1} \left(-\frac{u}{4\beta}\right)^k$$

and the density satisfies

$$(6) \quad \frac{\partial}{\partial \rho_{ij}} f_{n,p} = \frac{1}{2\beta} \sum_{m=0}^p \frac{p!}{m!} \prod_{i=m}^{p-1} \left(\frac{n}{2} + i\right)^{-1} \frac{\partial^2}{\partial x_i \partial x_j} f_{n,m}.$$

Proof. See the appendix.

Now define  $F_2$  and  $G_2$  as in (2) of Section 2. The next theorem generalizes Corollary 1. The proof consists of two steps: first, use (6) to establish the theorem for such densities, and second use a geometrical argument and (3) to show that the theorem is true for all elliptically contoured distributions.

Theorem 3. Corollary 1 is true for all bivariate elliptically contoured distributions.

Proof. Step 1. Using (5), we get

$$\frac{d}{d\rho} F_2 = \frac{1}{2\beta} \sum_{m=0}^p \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{2,m}(t; \frac{1}{\beta}, \beta) dt = \frac{1}{2\beta} \sum_{m=0}^p f_{2,m}(x; \frac{1}{\beta}, \beta)$$

and

$$\begin{aligned} \frac{d^2}{d\rho^2} F_2 &= \frac{1}{4\beta^2} \sum_{k=0}^p (p-k+1) \frac{\partial^2}{\partial x_1 \partial x_2} f_{2,k}(x; \frac{1}{\beta}, \beta) \\ &= \left\{ \frac{(\rho x_1 - x_2)(\rho x_2 - x_1)}{(1-\rho^2)^2} f_{2,p}(x; \frac{1}{\beta}, \beta) + \frac{\rho}{1-\rho^2} \frac{d}{d\rho} F_2 \right\}. \end{aligned}$$

Similarly, writing  $\tilde{x} = (x_1, -x_2)$  we have

$$\frac{d}{d\rho} G_2 = \frac{1}{\beta} \sum_{m=0}^p [f_{2,m}(x; \frac{1}{\beta}, \beta) - f_{2,m}(\tilde{x}; \frac{1}{\beta}, \beta)]$$

and

$$\begin{aligned} \frac{d^2}{d\rho^2} G_2 &= \frac{2(\rho x_1 - x_2)(\rho x_2 - x_1)}{(1-\rho^2)^2} f_{2,p}(x; \frac{1}{\beta}, \beta) + \frac{2(\rho x_1 + x_2)(\rho x_2 + x_1)}{(1-\rho^2)^2} f_{2,p}(\tilde{x}; \frac{1}{\beta}, \beta) \\ &\quad + \frac{\rho}{1-\rho^2} \frac{d}{d\rho} G_2. \end{aligned}$$



It is well known that  $\frac{d}{d\rho} F_2$  and  $\rho \frac{d}{d\rho} G_2$  are non-negative (Tong [8], p. 64, ff.). Thus, the theorem follows for this case immediately upon inspection of the other terms in the expression for  $\frac{d^2}{d\rho^2} F_2$  and  $\frac{d^2}{d\rho^2} G_2$ , respectively. (The use of (4) instead of (5) yields the same expressions for  $\frac{d^2}{d\rho^2} F_2$  and  $\frac{d^2}{d\rho^2} G_2$ .)

Step 2. If we let  $m = \frac{x_1}{x_2} \wedge \frac{x_2}{x_1}$  and assume  $x_1 x_2 > 0$ , then  $F_2$  is convex in  $\rho \in (0, m)$  whenever the density of  $X$  is  $f_{2,p}$ . Recall that  $X = R\frac{1}{2}U$ , where  $U$  is uniform on the unit circle, and  $R^2$  has density

$$h_{p,\beta}(r) = \frac{\beta^p}{p!} r^p e^{-\beta r}.$$

If  $e_1$  and  $e_2$  are the standard unit vectors, and  $b_i = \frac{1}{2}e_i$ , then

$$\begin{aligned} F_2(x; \rho) &= \int_0^\infty P(b_1'U \leq \frac{x_1}{r}, b_2'U \leq \frac{x_2}{r}) h_{p,\beta}(r) dr \\ &= \int_0^\infty \mu(x, r; \rho) h_{p,\beta}(r) dr. \end{aligned}$$

Since  $F_2$  is convex in  $\rho \in (0, m)$ , we have for  $0 \leq \alpha, \beta$  and  $\alpha + \beta = 1$ ,

$$0 \geq \int_0^\infty \{ \mu(x, r; \alpha\rho_1 + \beta\rho_2) - \alpha\mu(x, r; \rho_1) - \beta\mu(x, r; \rho_2) \} h_{p,\beta}(r) dr$$

for  $\rho_1, \rho_2 \in (0, m)$  and all  $p, \beta > 0$ . Now let  $p, \beta \rightarrow \infty$  with  $p/\beta = r_0^2$ ; it is easy to see that  $\mu(x, r_0, \rho)$  is convex in  $\rho \in (0, m)$  for each  $r_0$ . But since  $U$  is uniformly distributed,  $\mu$  is just proportional to the

lengths of certain arcs on the circle, so we now have the purely geometrical fact that this length is a convex function of  $\rho$ . Thus, the distribution of  $R$  is irrelevant and we have that  $F_2$  is convex in  $\rho \in (0, m)$  when  $x_1 x_2 > 0$  for all elliptically contoured distributions. The proof for the other cases is the same.

The key to Step 2 above is that for the family  $f_{n,p}(x; \dagger, \beta)$ , the radius vector  $\|\dagger^{-1/2} X\|$  can be made to degenerate to any point  $\gamma_0 \geq 0$  by suitable choice of  $p$  and  $\beta$ . In particular, this family of densities is not multivariate unimodal.

#### 4. Applications.

Two well-known probability inequalities are

Slepian's inequality: Let  $X$  and  $Y$  be elliptically contoured with  $EX = EY = 0$ , with parameters  $\dagger$  and  $T$ , respectively, where  $\dagger = (\rho_{ij})$ ,  $T = (\tau_{ij})$  are correlation matrices and  $\rho_{ij} \geq \tau_{ij}$ . Then for all  $a = (a_1, \dots, a_n)$ ,  $P(X_i \leq a_i; i=1, \dots, n) \geq P(Y_i \leq a_i; i=1, \dots, n)$ .

Dunn's inequality: Let  $X \sim N(0, \dagger)$ . Then  $P(|X_i| \leq a_i; i=1, \dots, n) \geq \prod_1^n P(|X_i| \leq a_i)$ .

Of particular interest for the construction of simultaneous confidence bounds is the case  $a_1 = a_2 = \dots = a_n = a$ . These inequalities have been used extensively to provide bounds for the probabilities of interest. We shall show below, however, that the bounds are often far from the true probabilities; that is, the bounds are bad approximations. We also show how convexity results from Sections 2 and 3 can yield very good approximations (but not necessarily bounds) for high dimensional probabilities.

Suppose then that  $X$  is elliptically contoured, and we want to approximate  $P(X_i \leq a; i=1, \dots, n)$  and  $P(|X_i| \leq a; i=1, \dots, n)$ . Let  $S_n = \sum_1^n I\{X_i \leq a\}$  and  $T_n = \sum_1^n I\{|X_i| \leq a\}$ , so the above probabilities are  $P(S_n = n)$  and  $P(T_n = n)$ , respectively. Also, let  $\bar{\mathbb{I}}$  be the equicorrelation matrix whose off-diagonal elements are all equal to the average of the off-diagonal terms of  $\mathbb{I}$ . Then we have

Theorem 4. If  $\rho_{ij} \geq 0$  for all  $i, j$ , then  $\text{var}_{\mathbb{I}} S_n \geq \text{var}_{\bar{\mathbb{I}}} S_n$ . For all  $\mathbb{I}$ ,  $\text{var}_{\Sigma} T_n \geq \text{var}_{\bar{\mathbb{I}}} T_n$ .

Proof. The variance of  $T_n$  is

$$\text{var}_{\mathbb{I}} T_n = \text{constant} + 2 \sum_{i < j} P(|X_i| \leq a, |X_j| \leq a) = k(\{\rho_{ij}\}).$$

Now  $P(|X_i| \leq a, |X_j| \leq a)$  is a convex function of  $\rho_{ij}$  by Theorem 3, so the Hessian of  $k$  is a positive definite (diagonal) matrix, and  $k$  is a convex function. Being a symmetric function,  $k$  is also Schur-convex; thus  $k(\{\rho_{ij}\}) \geq k(\{\bar{\rho}\})$ , where  $\bar{\rho} = \sum_{i < j} \rho_{ij} / \binom{n}{2}$ . The proof for  $S_n$  is similar.

Notice that  $E_{\mathbb{I}} S_n = E_{\bar{\mathbb{I}}} S_n$ , so that  $S_n$  has the same location under  $\mathbb{I}$  and  $\bar{\mathbb{I}}$ , but is more dispersed under  $\mathbb{I}$  when  $\rho_{ij} \geq 0$ . It is tempting to conjecture, then, that certain probability inequalities obtain for  $P(S_n = n)$ . For an illustration, suppose that  $a = 0$ , so that  $E_{\mathbb{I}} S_n = E_{\bar{\mathbb{I}}} S_n = n/2$ ; in this case, we might conjecture that  $P_{\mathbb{I}}(S_n = n) \geq P_{\bar{\mathbb{I}}}(S_n = n)$ , for  $\rho_{ij} \geq 0$ . Indeed, this is true for some special cases, but not in general. Even when the inequality does not hold, we can expect that approximating  $P_{\mathbb{I}}(S_n = n)$  by  $P_{\bar{\mathbb{I}}}(S_n = n)$  will be better than using the Slepian bounds,  $P_{\mathbb{I}_{\min}}(S_n = n)$  or

$P_{\dagger_{\max}}(S_n=n)$ , where  $\dagger_{\max}$  and  $\dagger_{\min}$  are equicorrelation matrices with parameters  $\max \rho_{ij}$  and  $\min \rho_{ij}$ , respectively. Table 1 below compares the two methods; for the correlation matrices used for comparison, see the Appendix.

TABLE 1

	True	$\dagger$ approx.	Slepian upper bound	Slepian lower bound
a)	.216	.214	.272	.164
b)	.134	.135	.228	.106
c)	.243	.241	.183	.273
d)	.129	.126	.008	.316

Of course, similar comments apply for  $a \neq 0$  and  $P_{\dagger}(T_n=n)$ . Also, we could use the concavity results of the earlier sections to give simple sufficient conditions for the opposite inequalities; we omit the details.

##### 5. Computations for the Equicorrelation Case.

The above approximations are most useful if they are easily evaluated; that is, the approximation be expressible as a low dimensional integral, which then requires numerical quadrature.

For the normal distribution, the equicorrelated case is especially easy. When  $\rho \geq 0$ , it is known that a one-dimensional integral suffices:

$$(6) \quad P(X_1 \leq a, \dots, X_n \leq a) = \int_{-\infty}^{\infty} \phi\left(\frac{a - \sqrt{\rho} t}{\sqrt{1-\rho}}\right)^n \phi(t) dt$$

where  $\phi$  and  $\Phi$  are the standard normal density and distribution, respectively. And for  $\rho < 0$ , Steck [7] provides a formula relating it to the  $\rho > 0$  case.

For other elliptically contoured distributions, the equicorrelated case does not yield a one-dimensional integral. Even when  $\rho = 0$ , the probability integrals do not immediately reduce in dimension because  $\rho = 0$  implies independence (within the elliptically contoured family) only for the normal distribution. For one subclass, though, a two-dimensional representation is possible. Cambanis, et al. [1] use a result of Schoenberg and von Neumann to show that if  $X$  has an elliptically contoured distribution, which can be embedded in an infinite exchangeable sequence, then  $X$  has density  $|\Sigma|^{-1/2} f(x' \Sigma^{-1} x)$  where  $(\Sigma)_{ij} = \rho$  for  $i \neq j$  and

$$(7) \quad f(u) = \int_{[0, \infty)} (2\pi r^2)^{-n/2} \exp(-u/2r^2) dF(r)$$

and  $F$  is the distribution of  $\|\Sigma^{-1/2} X\|$ . Using this result, we can apply Fubini's theorem to immediately get that

$$\begin{aligned} P(X_i \leq a; i=1, \dots, n) \\ = \int_{[0, \infty)} \left\{ \int_{-\infty}^a \dots \int_{-\infty}^a |2\pi r^2 \Sigma|^{-1/2} \exp\left(-\frac{1}{2} x' (r^2 \Sigma)^{-1} x\right) dx \right\} dF(r) . \end{aligned}$$

The inner integral is a Gaussian integral, which can be evaluated easily by (6); thus, we have that for scale mixtures of normals, orthant probabilities and rectangular probabilities are easily computed. Important special cases are the multivariate  $t$  and Cauchy distributions which are elliptically contoured.

For other elliptically contoured distributions, the problem is harder; it essentially reduces to that of computing the surface area of the intersection of circular patches on the sphere, since

$$P(X_1 \leq a_1; \mathbf{v}_1) = \int_{[0, \infty)} P(b_1' U \leq \frac{a_1}{r}; \mathbf{v}_1) dF(r) .$$

## 6. Extensions to Multivariate Case.

There are several ways in which to extend our convexity results to higher dimensions. For instance, one might consider  $F(\mathbf{A}; \mathbf{a}) = P_{\mathbf{A}}(X_1 \leq a_1; \mathbf{v}_1)$  and ask when  $F$  is a convex function of  $\mathbf{A}$ . A simpler but instructive case is when  $\mathbf{A} = \mathbf{A}_\rho$ , an equicorrelation matrix, and  $a_1 \equiv a$ . Even in this case, the computations are complicated, so our results are somewhat limited.

Theorem 5. If  $X \sim N_n(0, \mathbf{A}_\rho)$ , then  $P(X_1 \leq a; \mathbf{v}_1)$  is a convex function of  $\rho \geq 0$  when  $a \leq 0$ . When  $a \leq 1 - \sqrt{2}$ , it is a convex function of  $\rho$  in its entire domain  $(-\frac{1}{n-1}, 1)$ .

Proof. Let  $(dx)_n = dx_1, \dots, dx_n$  and interpret a 0-fold integral as 0.

Now

$$F(\rho; a) = P(X_1 \leq a; \mathbf{v}_1) = \int_{-\infty}^a \phi_n(x; \rho) (dx)_n$$

and



$$\begin{aligned}\frac{dF(\rho; a)}{d\rho} &= \binom{n}{2} \int_{-\infty}^a \phi_n(x, a, a; \rho) (dx)_{n-2} \\ &= \binom{n}{2} \phi_2(a, a; \rho) \int_{-\infty}^{at(\rho)} \phi_{n-2}(x; \rho') (dx)_{n-2}\end{aligned}$$

where  $t(\rho) = \{(1-\rho)/(1+\rho)(1+2\rho)\}^{1/2}$  and  $\rho' = \rho/(1+2\rho)$ . Next

$$\begin{aligned}\frac{d^2 F(\rho; a)}{d\rho^2} &= \binom{n}{2} \left\{ \left[ \frac{d}{d\rho} \phi_2(a, a; \rho) \right] \int_{-\infty}^{at(\rho)} \phi_{n-2}(x; \rho') (dx)_{n-2} \right. \\ &\quad \left. + \phi_2(a, a; \rho) \frac{d}{d\rho} \int_{-\infty}^{at(\rho)} \phi_{n-2}(x; \rho') (dx)_{n-2} \right\} .\end{aligned}$$

By Theorem 1, the first term is positive if  $|a| > \sqrt{2} - 1$  (for all  $\rho$ ) or  $\rho \geq 0$  (for all  $a$ ). Thus, we only need to study

$$\frac{d}{d\rho} \int_{-\infty}^{at(\rho)} \phi_{n-2}(x; \rho') (dx)_{n-2} .$$

Now recall that if  $G(x) = \int_{-\infty}^{b(x)} g(x, y) dy$ , then

$$G'(x) = \int_{-\infty}^{b(x)} g_x(x, y) dy + b'(x) g(x, b(x)) .$$

Repeated use of it shows that

$$\begin{aligned}\frac{d}{d\rho} \int_{-\infty}^{at(\rho)} \phi_{n-2}(x; \rho') (dx)_{n-2} \\ &= \frac{1}{(1+2\rho)^2} \binom{n-2}{2} \int_{-\infty}^{at(\rho)} \phi_{n-4}(x, at(\rho), at(\rho); \rho') (dx)_{n-4} \\ &\quad + (n-2) at'(\rho) \int_{-\infty}^{at(\rho)} \phi_{n-3}(x, at(\rho); \rho') (dx)_{n-3} .\end{aligned}$$

The first term here is always positive; the second term is positive only when  $a \leq 0$ , since  $t'(\rho) = -t(\rho)\left\{\frac{1}{1-\rho^2} + \frac{1}{1-\rho^2}\right\} \leq 0$ . Thus, we have the simple sufficient conditions for the positivity of  $d^2F(\rho;a)/d\rho^2$ .

Corollary 2. If  $X$  is a scale mixture of equicorrelated normals, then  $P(X_1 \leq a; \Psi_1)$  is convex in  $\rho \geq 0$  for  $a \leq 0$ .

Proof. From (7), we get

$$P(X_1 \leq a; \Psi_1) = \int_{[0, \infty)} P(Z_1 \leq ar; \Psi_1) dF(r)$$

where  $Z = (Z_1, \dots, Z_n) \sim N_n(0, \frac{1}{\rho})$ . Since  $a \leq 0 \Rightarrow ar \leq 0$ , the integrand is convex for  $\rho \geq 0$  for each  $r$ . Hence, the integral is also convex.

Note that this corollary indicates that the rather specialized result for the normal ( $a < 1 - \sqrt{2} \Rightarrow F(\rho;a)$  convex for all  $\rho$ ) cannot hold for all elliptically contoured distributions, since we may have  $ar > 1 - \sqrt{2}$  even though  $a < 1 - \sqrt{2}$ .

We have only succeeded in extending Corollary 2 to arbitrary elliptically contoured distributions when  $n=3$ , by using the family of distributions (5). If we let

$$F(\rho;a) = \int_{-\infty}^a f_{n,p}(x; \frac{1}{\rho}, \beta) (dx)_n ,$$

then

$$\frac{dF(\rho;a)}{d\rho} = \binom{n}{2} \frac{1}{2\beta} \sum_{m=0}^p \alpha_{n,m,p} \int_{-\infty}^a f_{n,m}(x, a, a; \frac{1}{\rho}, \beta) (dx)_{n-2}$$

where  $\alpha_{n,m,p} = \frac{p!}{m!} \prod_{i=m}^{p-1} \left(\frac{n}{2} + i\right)^{-1}$ . See the "working rules" for this family of distributions in the Appendix; the notation below is explained there.

Next, after some more algebra,

$$\begin{aligned} \frac{d^2}{d\rho^2} F(\rho; a) &= \binom{n}{2} \frac{1}{4\beta} \int_{-\infty}^a (\beta A_{n-1} A_n f_{n,p}^{-A_{n,n-1}} \sum_{k=0}^p \alpha_{n,k,p} f_{n,k}) \Big|_{(x,a,a;\frac{1}{\beta},\beta)} (dx)^{n-2} \\ &+ \binom{n}{2} \binom{n-2}{2} \frac{1}{4\beta^2} \sum_{k=0}^p (p-k+1) \alpha_{n,k,p} \int_{-\infty}^a f_{n,k}(x,a,a,a;\frac{1}{\beta},\beta) (dx)^{n-4} \\ &- \binom{n}{3} \left(\frac{3}{4\beta}\right) \sum_{k=0}^p \alpha_{n,k,p} \int_{-\infty}^a A_n f_{n,k} \Big|_{(x,a,a,a;\frac{1}{\beta},\beta)} (dx)^{n-3} . \end{aligned}$$

The second term is obviously positive. The first term is positive when  $\rho \geq 0$  because

$$A_{n,n,-1} = - \frac{2\rho}{(1-\rho)(1+(n-1)\rho)} \leq 0 ,$$

and

$$A_{n-1} A_n = \frac{4}{(1-\rho)^2 (1+(n-1)\rho)^2} (a(1+(n-3)\rho) - \rho \sum_{i=1}^{n-2} x_i)^2 \geq 0 .$$

The sign of the third term is largely determined by  $A_n$ ; here

$$A_n = \frac{2}{(1-\rho)(1+(n-1)\rho)} (a(1+(n-4)\rho) - \rho \sum_{i=1}^{n-3} x_i) .$$

Thus, when  $n=3$ ,  $A_n$  is negative for  $a \leq 0$ , and so Corollary 2 extends to all elliptically contoured distributions (by the same geometrical argument of Theorem 3). For  $n \geq 4$ ,  $A_n < 0$  and it is no longer easy to see where  $\frac{d^2}{d\rho^2} F(\rho; a)$  is positive.

Appendix.

(i) Proof of Lemma 2. Let  $\Omega_n(|s|^2) = E \exp(is'U)$  be the characteristic function of the uniform distribution on the unit sphere in  $R^n$ . It is known from the Gaussian case that

$$\begin{aligned} e^{-u/2} &= \int_0^\infty \Omega_n(r^2 u) \frac{r^{n-1} e^{-r^2/2}}{2^{n/2-1} \Gamma(\frac{n}{2})} dr \\ &= \int_0^\infty \Omega_n(r^2) \frac{r^{n-1} e^{-r^2/2u}}{2^{n/2-1} \Gamma(\frac{n}{2})} \frac{dr}{u^{n/2}}. \end{aligned}$$

Differentiating both sides with respect to  $u$  and rearranging some terms, we get

$$\int_0^\infty \Omega_n(r^2) \frac{r^{n+2k-1} e^{-r^2/2u}}{\Gamma(\frac{n}{2}) 2^{n/2-1}} dr = e^{-u/2} u^{n/2+k} \sum_{i=0}^k \binom{k}{i} \prod_{j=0}^{k-1} (n+2j) (-u)^i.$$

Now recall that if  $X$  has density  $f_{n,p}(x; \frac{1}{2}, \beta)$ , then  $X \stackrel{d}{=} R \frac{1}{2} U$ , where  $R^2$  has density

$$h_{p,\beta}(r;n) = \frac{\beta^{p+\frac{n}{2}}}{\Gamma(p+\frac{n}{2})} r^{p+\frac{n}{2}-1} e^{-\beta r}$$

so that the characteristic function of  $X$  is given by  $\hat{f}(t) = \phi_{n,p}(t' \frac{1}{2} t; \beta)$ , where

$$\begin{aligned} \phi_{n,p}(u; \beta) &= \int_0^\infty \Omega_n(r^2 u) h_{p,\beta}(r^2; n) 2r dr \\ &= \int_0^\infty \Omega_n(r^2) h_{p,\beta}(\frac{r^2}{u}; n) \frac{2r dr}{u} \end{aligned}$$

is given in the lemma.

Next, to get the generalization of Plackett's identity, recall that the

Fourier inversion formula gives

$$f_{n,p}(x; \frac{1}{4}, \beta) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix't} \phi_{n,p}(t; \frac{1}{4}t; \beta) dt.$$

Differentiating under the integral, we get

$$\frac{\partial f_{n,p}}{\partial x_{ij}} = -\frac{1}{2\beta} \int_{\mathbb{R}^n} e^{-ix't} e^{-\frac{t' \frac{1}{4} t}{4\beta}} t_i t_j \sum_{k=0}^p \binom{p}{k} \frac{p + \frac{n}{2}}{\prod_{v=0}^k (\frac{n}{2} + v)} \left(-\frac{t' \frac{1}{4} t}{4\beta}\right)^k dt$$

and

$$\frac{\partial^2 f_{n,p}}{\partial x_i \partial x_j} = - \int_{\mathbb{R}^n} e^{-ix't} e^{-\frac{t' \frac{1}{4} t}{4\beta}} t_i t_j \sum_{k=0}^p \binom{p}{k} \prod_{v=0}^{k-1} \left(\frac{n}{2} + v\right)^{-1} \left(-\frac{t' \frac{1}{4} t}{4\beta}\right)^k dt.$$

Thus, we get the partial differential equation because the coefficients are equal:

$$\frac{p + \frac{n}{2}}{\prod_{i=0}^k (\frac{n}{2} + i)} \binom{p}{k} = \sum_{m=k}^p \frac{p!}{m!} \binom{m}{k} \prod_{j=m}^{p-1} \left(\frac{n}{2} + j\right)^{-1} \prod_{i=1}^{k-1} \left(\frac{n}{2} + i\right)^{-1}$$

which can be established by induction. An empty product is to be interpreted as 1.

ii) For convenience, we list some "working rules" for the family of densities  $f_{n,p}(x; \frac{1}{4}, \beta)$ . Write

$$A = x' \frac{1}{4}^{-1} x, \quad C_p = 2\beta^{p + \frac{n}{2}} / s_n \Gamma(\frac{n}{2} + p) \left|\frac{1}{4}\right|^{1/2}$$

and

$$A_i = \frac{\partial}{\partial x_i} A, \quad A_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} A,$$

where  $s_n$  is the surface area of the unit sphere in  $R^n$ .

$$a) \quad f_{n,p}(x; \frac{1}{2}, \beta) = C_p A^p e^{-\beta A}$$

$$b) \quad f_{n,p}/f_{n,p-1} = \beta A / (\frac{n}{2} + p - 1)$$

$$c) \quad \frac{\partial}{\partial x_i} f_{n,p} = \beta A_i \left( \frac{p}{\frac{n}{2} + p - 1} f_{n,p-1} - f_{n,p} \right)$$

$$d) \quad \frac{\partial^2}{\partial x_i \partial x_j} f_{n,p} = \beta \{ (\beta A_i A_j - A_{ij}) f_{n,p} - \frac{p}{\frac{n}{2} + p - 1} (2\beta A_i A_j - A_{ij}) f_{n,p-1} \\ + \beta A_i A_j \frac{p(p-1)}{(\frac{n}{2} + p - 1)(\frac{n}{2} + p - 2)} f_{n,p-2} \}$$

e) If, in Lemma 2, we write

$$\frac{\partial f_{n,p}}{\partial \rho_{ij}} = \frac{1}{2\beta} \sum_{m=0}^p \alpha_{n,m,p} \frac{\partial^2}{\partial x_i \partial x_j} f_{n,m}$$

then

$$\alpha_{n,m,p} = \frac{p!}{m!} \prod_{i=m}^{p-1} \left( \frac{n}{2} + i \right)^{-1}$$

satisfies

$$\alpha_{n,m,p} \alpha_{n,k,m} = \alpha_{n,k,p} \quad \text{and} \quad \alpha_{2,m,p} \equiv 1.$$

iii) The correlation matrices used to get the entries in Table 1 are the following.



$$\text{a)} \begin{bmatrix} 1.000 & 0.600 & 0.636 & 0.472 \\ & 1.000 & 0.383 & 0.458 \\ & & 1.000 & 0.705 \\ & & & 1.000 \end{bmatrix}$$

$$\text{b)} \begin{bmatrix} 1.000 & 0.585 & 0.177 & 0.197 \\ & 1.000 & 0.209 & 0.217 \\ & & 1.000 & 0.291 \\ & & & 1.000 \end{bmatrix}$$

$$\text{c)} \begin{bmatrix} 1.000 & 0.579 & 0.241 \\ & 1.000 & 0.582 \\ & & 1.000 \end{bmatrix}$$

$$\text{d)} \begin{bmatrix} 1.000 & 0.717 & -0.232 \\ & 1.000 & -0.471 \\ & & 1.000 \end{bmatrix}$$

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